

# Maximal $m$ -distance sets containing the representation of the Hamming graph $H(n, m)$

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## Abstract

A set  $X$  in the Euclidean space  $\mathbb{R}^d$  is an  $m$ -distance set if the set of Euclidean distances between two distinct points in  $X$  has size  $m$ . An  $m$ -distance set  $X$  in  $\mathbb{R}^d$  is *maximal* if there does not exist a vector  $\mathbf{x}$  in  $\mathbb{R}^d$  such that the union of  $X$  and  $\{\mathbf{x}\}$  still has only  $m$  distances. Bannai, Sato, and Shigezumi (2012) investigated maximal  $m$ -distance sets that contain the Euclidean representation of the Johnson graph  $J(n, m)$ . In this paper, we consider the same problem for the Hamming graph  $H(n, m)$ . The Euclidean representation of  $H(n, m)$  is an  $m$ -distance set in  $\mathbb{R}^{m(n-1)}$ . We prove that if the representation of  $H(n, m)$  is not maximal as an  $m$ -distance set for some  $m$ , then the maximum value of  $n$  is  $m^2 + m - 1$ . Moreover we classify the largest  $m$ -distance sets that contain the representation of  $H(n, m)$  for  $n \geq 2$  and  $m \leq 4$ . We also classify the maximal 2-distance sets that are in  $\mathbb{R}^{2n-1}$  and contain the representation of  $H(n, 2)$  for  $n \geq 2$ .

**Key words:** Hamming graph, few-distance set, Euclidean representation, Erdős–Ko–Rado theorem.

## 1 Introduction

A subset  $X$  of the Euclidean space  $\mathbb{R}^d$  is an  $m$ -distance set if the size of the set of distances between two distinct points in  $X$  is equal to  $m$ . The size of an  $m$ -distance set is bounded above by  $\binom{d+m}{m}$  [3]. One of major problems is to find the maximum possible cardinality of an  $m$ -distance set for given  $m$  and  $d$ . The largest 1-distance set in  $\mathbb{R}^d$  is the regular simplex for  $d \geq 1$ , and it has  $d + 1$  points. Largest 2-distance sets in  $\mathbb{R}^d$  are classified for  $d \leq 7$  [8, 11]. Lisoněk [11] constructed a largest 2-distance set in  $\mathbb{R}^8$ , which is the only known set attaining the bound  $|X| \leq \binom{d+m}{m}$  for  $m \geq 2$ . Largest  $m$ -distance sets in  $\mathbb{R}^2$  are classified for  $m \leq 5$  [6, 12, 13]. Two largest 6-distance sets are known [15]. Tables 1, 2 show the cardinalities  $|X|$  of largest distance sets  $X$ , and the number  $\#$  of the sets, up to isometry. The largest

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Table 1: $m = 2$							
$d$	2	3	4	5	6	7	8
$ X $	5	6	10	16	27	29	45
#	1	6	1	1	1	1	$\geq 1$

Table 2: $d = 2$					
$m$	2	3	4	5	6
$ X $	5	7	9	12	13
#	1	2	4	1	$\geq 2$

3-distance set in  $\mathbb{R}^3$  is the vertex set of the icosahedron [14].

The Euclidean representation  $\tilde{J}(n, m)$  of the Johnson scheme  $J(n, m)$  is the subset of  $\mathbb{R}^n$  consisting of all vectors with 1's in  $m$  coordinates and 0's elsewhere. The set  $\tilde{J}(n, m)$  with  $n \geq 2m$  can be interpreted as an  $m$ -distance set in  $\mathbb{R}^{n-1}$  because the sum of entries of each element is  $m$ . The largest known  $m$ -distance sets in  $\mathbb{R}^{n-1}$  are mostly  $\tilde{J}(n, m)$ . An  $m$ -distance set  $X$  in  $\mathbb{R}^n$  is *maximal* if there does not exist  $\mathbf{x} \in \mathbb{R}^n$  such that  $X \cup \{\mathbf{x}\}$  is still  $m$ -distance. Bannai, Sato, and Shigezumi [4] investigated maximal  $m$ -distance sets that are in  $\mathbb{R}^{n-1}$  and contain  $\tilde{J}(n, m)$ . They gave a necessary and sufficient condition for  $\tilde{J}(n, m)$  to be a maximal  $m$ -distance set in  $\mathbb{R}^{n-1}$ , and classified the largest  $m$ -distance sets containing  $\tilde{J}(n, m)$  for  $n \geq 2$  and  $m \leq 5$ , except for  $(n, m) = (9, 4)$ . The case  $(n, m) = (9, 4)$  is solved in [1]. This construction of distance sets might be possible for other association schemes. In this paper we consider the Hamming scheme  $H(n, m)$ .

Let  $F_n = \{1, \dots, n\}$ ,  $\mathbf{x} = (x_1, \dots, x_m) \in F_n^m$ , and  $\mathbf{y} = (y_1, \dots, y_m) \in F_n^m$ . The Hamming distance of  $\mathbf{x}$  and  $\mathbf{y}$  is defined to be  $h(\mathbf{x}, \mathbf{y}) = |\{i: x_i \neq y_i\}|$ . The Hamming scheme  $H(n, m)$  is an association scheme  $(F_n^m, \{R_0, \dots, R_m\})$ , where  $R_i = \{(\mathbf{x}, \mathbf{y}): h(\mathbf{x}, \mathbf{y}) = i\}$ . Let  $\varphi: F_n^m \rightarrow \mathbb{R}^{mn}$  be the embedding defined by

$$\varphi: \mathbf{x} = (x_1, \dots, x_m) \mapsto \tilde{\mathbf{x}} = \sum_{i=1}^m \mathbf{e}_{(i-1)n+x_i},$$

where  $\{\mathbf{e}_1, \dots, \mathbf{e}_{mn}\}$  is the standard basis of  $\mathbb{R}^{mn}$ . Let  $\tilde{H}(n, m)$  denote the image of  $\varphi$ . Note that  $h(\mathbf{x}, \mathbf{y}) = k$  for  $\mathbf{x}, \mathbf{y} \in H(n, m)$  if and only if  $d(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \sqrt{2k}$  for  $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \tilde{H}(n, m)$ , where  $d(\cdot, \cdot)$  is the Euclidean distance. Let  $\mathbf{j}_k$  denote the vector

$$\mathbf{j}_k = \sum_{i=(k-1)n+1}^{kn} \mathbf{e}_i.$$

Every vector in  $\tilde{H}(n, m)$  is perpendicular to  $\mathbf{j}_k$  for  $k \in \{1, \dots, m\}$ . We can therefore interpret  $\tilde{H}(n, m)$  as an  $m$ -distance set in  $\mathbb{R}^{m(n-1)}$ . We consider maximal  $m$ -distance sets that are in  $\mathbb{R}^{m(n-1)}$  and contain  $\tilde{H}(n, m)$ .

This paper is summarized as follows. In Section 2, we give some notation, and determine the coordinates of a vector  $\mathbf{x}$  when  $\mathbf{x}$  can be added to  $\tilde{H}(n, m)$  while maintaining  $m$ -distance. In Section 3, the maximal 2-distance sets containing  $\tilde{H}(n, 2)$  are classified by an explicit way. In Section 4, we give a necessary and sufficient condition for  $\tilde{H}(n, m)$  to be maximal as an  $m$ -distance set. Moreover, we prove that if  $\tilde{H}(n, m)$  is not maximal as an  $m$ -distance set for

Table 3:  $m = 2$ 

$n$	5
$d$	8
$ X $	40

Table 4:  $m = 3$ 

$n$	3	5	9	11
$d$	6	12	24	30
$ X $	40	200	981	1451

Table 5:  $m = 4$ 

$n$	2	3	5	6	7	9	11	13	14	19
$d$	4	8	16	20	24	32	40	48	52	72
$ X $	25	222	1600	2004	3390	8829	16566	29056	39417	133381

some  $m$ , then the maximum value of  $n$  is equal to  $m^2 + m - 1$ . In Section 5, we classify the largest  $m$ -distance sets that are in  $\mathbb{R}^{m(n-1)}$  and contain  $\tilde{H}(n, m)$  for  $n \geq 2$  and  $m \leq 4$ . Tables 3–5 show the maximum cardinalities  $|X|$  and dimension  $d = m(n - 1)$ . In Section 6, we classify maximal 2-distance sets that are in  $\mathbb{R}^{2(n-1)+1}$  and contain  $\tilde{H}(n, 2)$ .

## 2 Vectors that can be added to $\tilde{H}(n, m)$

First we give some notation. For real numbers  $x_1, \dots, x_n$  and natural numbers  $\lambda_1, \dots, \lambda_n$ , we use the notation

$$(x_1^{\lambda_1}, \dots, x_n^{\lambda_n}) = (\underbrace{x_1, \dots, x_1}_{\lambda_1}, \dots, \underbrace{x_n, \dots, x_n}_{\lambda_n}) \in \mathbb{R}^{\lambda_1 + \dots + \lambda_n},$$

and  $(x_1^{\lambda_1})$  is abbreviated to  $x_1^{\lambda_1}$ . Let  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$  and  $X_i \subset \mathbb{R}^n$ . We use the following notation:

$$\mathbf{x}^P = (x_1, \dots, x_n)^P = \{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) : \sigma \in S_n\} \subset \mathbb{R}^n,$$

$$(X_1, \dots, X_m) = \{(\mathbf{x}_1, \dots, \mathbf{x}_m) : \mathbf{x}_i \in X_i\} \subset \mathbb{R}^{mn},$$

$$(X_1, \dots, X_m)^P = \{(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(m)}) : \mathbf{x}_i \in X_i, \sigma \in S_m\} \subset \mathbb{R}^{mn},$$

where  $S_n$  is the permutation group. A one-element set  $\{\mathbf{x}\}$  is abbreviated to  $\mathbf{x}$  in these expressions.

Suppose  $\mathbf{x} \in \mathbb{R}^{mn}$  can be added to  $\tilde{H}(n, m)$  while maintaining  $m$ -distance. Note that  $d(\mathbf{x}, \mathbf{y}) \in \{\sqrt{2}, \sqrt{4}, \dots, \sqrt{2m}\}$  for each  $\mathbf{y} \in \tilde{H}(n, m)$ . For  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_m)$  with  $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$ , the sum of all entries of  $\mathbf{x}_i$  is 1 for each  $i \in \{1, \dots, m\}$ . Because of the automorphism group of  $\tilde{H}(n, m)$ , each vector in  $(\mathbf{x}_1^P, \dots, \mathbf{x}_m^P)^P$  can be added to  $\tilde{H}(n, m)$ .

For  $\mathbf{x} = (x_1, \dots, x_{mn})$ , and  $\mathbf{y}, \mathbf{y}' \in \tilde{H}(n, m)$  such that  $\mathbf{y} = \sum_{i=1}^m \mathbf{e}_{(i-1)n+q_i}$  and  $\mathbf{y}' = \sum_{i=1}^m \mathbf{e}_{(i-1)n+q'_i}$ , we obtain

$$\sum_{i=1}^m (x_{(i-1)n+q_i} - x_{(i-1)n+q'_i}) \in \{0, \pm 1, \dots, \pm(m-1)\} \quad (2.1)$$

from  $d(\mathbf{x}, \mathbf{y})^2 - d(\mathbf{x}, \mathbf{y}')^2$ . Let  $j \in \{1, \dots, m\}$  be fixed. If  $q_j \neq q'_j$  and  $q_i = q'_i$  for  $i \neq j$ , then

$$x_{(j-1)n+q_j} - x_{(j-1)n+q'_j} \in \{0, \pm 1, \dots, \pm(m-1)\}.$$

Thus the block  $\mathbf{x}_j$  satisfies

$$\mathbf{x}_j \in (\alpha_j^{k_1^{(j)}}, (\alpha_j - 1)^{k_2^{(j)}}, \dots, (\alpha_j - t_j + 1)^{k_{t_j}^{(j)}})^P$$

for some  $\alpha_j \in \mathbb{R}$ ,  $t_j \in \mathbb{Z}$  with  $1 \leq t_j \leq m$ , and  $k_i^{(j)} \in \mathbb{Z}$  such that  $0 \leq k_i^{(j)}$  and  $\sum_{i=1}^{t_j} k_i^{(j)} = n$ . By (2.1) and  $1 \leq t_j \leq m$ , we have

$$\sum_{j=1}^m t_j \leq 2m - 1. \quad (2.2)$$

Since the sum of all entries of  $\mathbf{x}_j$  is 1, it follows that

$$n\alpha_j = 1 + \sum_{i=1}^{t_j} (i-1)k_i^{(j)}.$$

Let  $k_0^{(j)} = 1 + \sum_{i=1}^{t_j} (i-1)k_i^{(j)} \in \mathbb{Z}$ . Now we have

$$\mathbf{x}_j \in \left( \frac{k_0^{(j)}}{n}^{k_1^{(j)}}, \left( \frac{k_0^{(j)}}{n} - 1 \right)^{k_2^{(j)}}, \dots, \left( \frac{k_0^{(j)}}{n} - t_j + 1 \right)^{k_{t_j}^{(j)}} \right)^P.$$

For  $\mathbf{x}$  with  $\mathbf{x}_j = (x_1, \dots, x_n)$  and  $\mathbf{y}$  with  $\mathbf{y}_j = \mathbf{e}_{q_j}$ , let

$$l_i^{(j)} = \begin{cases} 1 & \text{if } x_{q_j} = k_0^{(j)}/n - i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The squared distance  $d(\mathbf{x}, \mathbf{y})^2$  can be expressed by

$$\begin{aligned} d(\mathbf{x}, \mathbf{y})^2 &= \sum_{j=1}^m \left( \sum_{i=1}^{t_j} l_i^{(j)} \left( 1 - \left( \frac{k_0^{(j)}}{n} - i + 1 \right) \right)^2 + \sum_{i=1}^{t_j} (k_i^{(j)} - l_i^{(j)}) \left( \frac{k_0^{(j)}}{n} - i + 1 \right)^2 \right) \\ &= \sum_{j=1}^m \left( 1 - n - 2k_0^{(j)} - \frac{k_0^{(j)2}}{n} + \sum_{i=1}^{t_j} i^2 k_i^{(j)} + 2 \sum_{i=1}^{t_j} i l_i^{(j)} \right). \end{aligned} \quad (2.3)$$

### 3 Case $m = 2$

In this section, we demonstrate a manner of classifying maximal 2-distance sets which contain  $\tilde{H}(n, 2)$  as easy case. Suppose  $\mathbf{x} \in \mathbb{R}^{2n}$  can be added to  $\tilde{H}(n, 2)$  while maintaining 2-distance. By (2.2), we have  $(t_1, t_2) = (1, 1), (2, 1)$ , or  $(1, 2)$ . Thus  $\mathbf{x}$  can be expressed by

$$\mathbf{x} \in \left( \left( \frac{k_0^{(1)}}{n}^{k_1^{(1)}}, \left( \frac{k_0^{(1)}}{n} - 1 \right)^{n-k_1^{(1)}} \right)^P, \frac{1}{n}^n \right)^P$$

such that  $k_0^{(1)} = 1 + n - k_1^{(1)}$ . By (2.3), for  $\mathbf{y} \in \tilde{H}(n, 2)$ , we have

$$d(\mathbf{x}, \mathbf{y})^2 = -1 - \frac{1}{n} + k_0^{(1)} - \frac{k_0^{(1)^2}}{n} + 2l_1^{(1)} + 4l_2^{(1)} \in \{2, 4\}.$$

This implies that

$$-1 - \frac{1}{n} + k_0^{(1)} - \frac{k_0^{(1)^2}}{n} = 0,$$

and hence  $n = k_0^{(1)} + 1 + 2/(k_0^{(1)} - 1)$ . The possible pairs are  $(n, k_0^{(1)}) = (5, 2), (5, 3)$ , namely,  $\mathbf{x}$  is contained in

$$X_1 = \left( \left( \frac{2^4}{5}, -\frac{3}{5} \right)^P, \frac{1^5}{5} \right)^P \text{ or } X_2 = \left( \left( \frac{3^3}{5}, -\frac{2^2}{5} \right)^P, \frac{1^5}{5} \right)^P.$$

Considering the distance of each pair of elements of  $X_1 \cup X_2$ , the maximal 2-distance set containing  $\tilde{H}(n, 2)$  is

$$\left( \left( \frac{2^4}{5}, -\frac{3}{5} \right)^P, \frac{1^5}{5} \right)^P \cup \left( \frac{1^5}{5}, \left( \frac{3^3}{5}, -\frac{2^2}{5} \right)^P \right)^P \cup \tilde{H}(n, 2) \quad [40 \text{ points}],$$

up to block permutations.

## 4 Conditions for the maximality of $\tilde{H}(n, m)$

In this section, we give a necessary and sufficient condition for  $\tilde{H}(n, m)$  to be a maximal  $m$ -distance set in  $\mathbb{R}^{n(m-1)}$ . We also show a manner of finding vectors that can be added to  $\tilde{H}(n, m)$ .

Let  $\mathfrak{x}_j$  denote the subset of  $\mathbb{R}^n$  defined by

$$\mathfrak{x}_j = \mathfrak{x}_j(k_0^{(j)}, \dots, k_{t_j}^{(j)}) = \left( \frac{k_0^{(j)}}{n}^{k_1^{(j)}}, \left( \frac{k_0^{(j)}}{n} - 1 \right)^{k_2^{(j)}}, \dots, \left( \frac{k_0^{(j)}}{n} - t_j + 1 \right)^{k_{t_j}^{(j)}} \right)^P$$

such that  $k_0^{(j)} = 1 + \sum_{i=1}^{t_j} (i-1)k_i^{(j)}$ . Let  $\mathfrak{X}$  denote the subset of  $\mathbb{R}^{mn}$  defined by

$$\mathfrak{X} = (\mathfrak{x}_1, \dots, \mathfrak{x}_m). \quad (4.1)$$

The integers  $k_i^{(j)}$  with  $1 \leq i \leq t_j$  and  $1 \leq j \leq m$  are called the parameters of  $\mathfrak{X}$ . If  $\mathbf{x} \in \mathbb{R}^{mn}$  can be added to  $\tilde{H}(n, m)$ , then  $\mathbf{x}$  is in  $\mathfrak{X}$  with some parameters  $k_i^{(j)}$ . Moreover if some vector in  $\mathfrak{X}$  can be added to  $\tilde{H}(n, m)$ , then so does every vector in  $\mathfrak{X}$ . For given  $\mathbf{x} \in \mathfrak{X}$ , let

$$M_{\mathfrak{X}} = \max_{\mathbf{y} \in \tilde{H}(n, m)} d(\mathbf{x}, \mathbf{y})^2.$$

Note that  $M_{\mathfrak{X}}$  does not depend on the choice of  $\mathbf{x} \in \mathfrak{X}$ . The value  $M_{\mathfrak{X}}$  determine if  $x \in \mathfrak{X}$  can be added to  $\tilde{H}(n, m)$ .

**Lemma 4.1.** *Each vector  $\mathbf{x} \in \mathfrak{X}$  can be added to  $\tilde{H}(n, m)$  while maintaining  $m$ -distance if and only if  $M_{\mathfrak{X}}$  is an even positive integer less than or equal to  $2m$ .*

*Proof.* Sufficiency is clear. Suppose  $M_{\mathfrak{X}}$  is an even positive integer less than or equal to  $2m$ . For any  $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}$  and  $\mathbf{y}, \mathbf{y}' \in \tilde{H}(n, m)$ , the difference between  $d(\mathbf{x}, \mathbf{y})^2$  and  $d(\mathbf{x}', \mathbf{y}')^2$  is an even integer by (2.3). Therefore,  $d(\mathbf{x}, \mathbf{y})^2$  is an even positive integer less than or equal to  $2m$  for any  $\mathbf{x} \in \mathfrak{X}$  and  $\mathbf{y} \in \tilde{H}(n, m)$ . This implies each  $\mathbf{x} \in \mathfrak{X}$  can be added to  $\tilde{H}(n, m)$ .  $\square$

It follows from (2.3) that

$$M_{\mathfrak{X}} = \sum_{j=1}^m \left( 1 - n - 2k_0^{(j)} - \frac{k_0^{(j)^2}}{n} + \sum_{i=1}^{t_j} i^2 k_i^{(j)} + 2t_j \right). \quad (4.2)$$

If  $t_l \geq 3$  for  $\mathfrak{X}$  with parameters  $k_i^{(j)}$ , then we can obtain  $\mathfrak{X}'$  with parameters  $k'_i{}^{(j)}$ , where  $k'_i{}^{(j)}$  are defined as follows:

$$\begin{aligned} k'_i{}^{(j)} &= k_i^{(j)} && \text{for } (j, i) \neq (l, 1), (l, 2), (l, t_l - 1), (l, t_l), \\ k'_1{}^{(l)} &= k_1^{(l)} - 1, & k'_2{}^{(l)} &= k_2^{(l)} + 1, & k'_{t_l-1}{}^{(l)} &= k_{t_l-1}^{(l)} + 1, & k'_{t_l}{}^{(l)} &= k_{t_l}^{(l)} - 1 && \text{if } t_l \geq 4, \\ k'_1{}^{(l)} &= k_1^{(l)} - 1, & k'_2{}^{(l)} &= k_2^{(l)} + 2, & k'_3{}^{(l)} &= k_3^{(l)} - 1 && \text{if } t_l = 3. \end{aligned}$$

**Lemma 4.2.** *If  $M_{\mathfrak{X}}$  is even, then  $M_{\mathfrak{X}'}$  is also even and*

$$M_{\mathfrak{X}} > M_{\mathfrak{X}'}.$$

*Proof.* It follows from (4.2) that  $M_{\mathfrak{X}} - M_{\mathfrak{X}'} = 2(2t_l - t'_l - 2)$ . Note that  $t'_l = t_l$  if  $k_{t_l}^{(l)} \geq 2$ , and  $t'_l = t_l - 1$  if  $k_{t_l}^{(l)} = 1$ . Since  $t_l \geq 3$  holds,  $M_{\mathfrak{X}} - M_{\mathfrak{X}'}$  is an even positive integer. This lemma therefore follows.  $\square$

**Remark 4.3.** By Lemmas 4.1 and 4.2, if each element of  $\mathfrak{X}$  can be added to  $\tilde{H}(n, m)$ , then so does each element of  $\mathfrak{X}'$ . Repeating this modification  $\mathfrak{X}'$ , we can obtain  $\mathfrak{X}_0$ , which satisfies  $t_j \leq 2$  for  $1 \leq j \leq m$  and each element of  $\mathfrak{X}_0$  can be added to  $\tilde{H}(n, m)$ .

**Lemma 4.4.** *The following are equivalent.*

- (1)  $\tilde{H}(n, m)$  is not maximal as an  $m$ -distance set.
- (2) There exist integers  $l, k_0^{(1)}, \dots, k_0^{(m)}$  such that  $n \geq k_0^{(1)} \geq \dots \geq k_0^{(l)} > 1 = k_0^{(l+1)} = \dots = k_0^{(m)}$ ,  $k_0^{(i)} \neq n$  for some  $i$ , and

$$\sum_{j=1}^m \frac{k_0^{(j)}(n - k_0^{(j)})}{n} + 2l \quad (4.3)$$

*is an even integer less than or equal to  $2m$ .*

*Proof.* Suppose  $\mathfrak{X}$  with parameters  $k_i^{(j)}$  satisfies  $t_1 = \dots = t_l = 2$  and  $t_{l+1} = \dots = t_m = 1$  for some  $l$ . From (4.2) and equations  $k_1^{(j)} = n + 1 - k_0^{(j)}$ ,  $k_2^{(j)} = k_0^{(j)} - 1$ , we have

$$M_{\mathfrak{X}} = \sum_{j=1}^m \frac{k_0^{(j)}(n - k_0^{(j)})}{n} + 2l.$$

By Lemma 4.1 and Remark 4.3, this lemma follows.  $\square$

**Remark 4.5.** By symmetry of (4.3), there exist  $l, k_0^{(1)}, \dots, k_0^{(m)}$  such that they satisfy the condition (2) in Lemma 4.4 if and only if there exist  $l, k_1, \dots, k_m$  such that  $0 \leq k_1 \leq \dots \leq k_l \leq n/2, k_{l+1} = \dots = k_m = 1, k_i \neq 0$  for some  $i$ , and

$$\sum_{j=1}^m \frac{k_j(n - k_j)}{n} + 2l$$

is an even integer less than or equal to  $2m$ .

**Theorem 4.6.** *If  $\tilde{H}(n, m)$  is not maximal as an  $m$ -distance set for some  $m$ , then the maximum value of  $n$  is  $m^2 + m - 1$ .*

*Proof.* Suppose  $\tilde{H}(n, m)$  is not maximal as an  $m$ -distance set. By Lemma 4.4 and Remark 4.5, there exist  $l, k_1, \dots, k_m$  such that  $0 \leq k_1 \leq \dots \leq k_l \leq n/2, k_{l+1} = \dots = k_m = 1, k_i \neq 0$  for some  $i$ , and

$$\sum_{j=1}^m k_j - \sum_{j=1}^m \frac{k_j^2}{n} + 2l$$

is an even integer at most  $2m$ . Let  $M = \sum_{j=1}^m k_j - \sum_{j=1}^m k_j^2/n + 2l$ , and  $t = \sum_{j=1}^m k_j^2/n$ . If  $l = 0$  holds, then  $M = m - m/n \in \mathbb{Z}$ . This implies  $n \leq m < m^2 + m - 1$ . Therefore we may suppose  $l \geq 1$ .

Assume  $t \geq 2m - 1$ . Since  $k_i \leq n/2$  for each  $i$ , we have  $M \geq t + 2l \geq 2m + 1$ , which contradicts  $M \leq 2m$ .

Assume  $t \leq 2m - 2$ . Since  $M \leq 2m$ , we have

$$\sum_{j=1}^l k_j \leq t + m - l.$$

It is therefore satisfied that

$$\begin{aligned} n &= \frac{1}{t} \sum_{j=1}^m k_j^2 = \frac{1}{t} \sum_{j=1}^l k_j^2 + \frac{m-l}{t} \leq \frac{1}{t} \left( \sum_{j=1}^l k_j \right)^2 + \frac{m-l}{t} \leq \frac{(t+m-l)^2}{t} + \frac{m-l}{t} \\ &\leq t + 2(m-1) + \frac{m(m-1)}{t} \leq \max\{m^2 + m - 1, \frac{9}{2}m - 4\} = m^2 + m - 1 \end{aligned}$$

for  $m \geq 2$  and  $1 \leq t \leq 2m - 2$ . Moreover for  $n = m^2 + m - 1, k_1 = m$ , and  $k_2 = \dots = k_m = 1$ , we have  $M = 2m$ . The theorem therefore follows.  $\square$

The remaining part of this section relates to the smallest value of  $m$  for fixed  $n, l, k_0^{(1)}, \dots, k_0^{(l)}$  in Lemma 4.4 (2). Each vector in  $\mathfrak{X}$  can be added to  $\tilde{H}(n, m)$  if and only if each vector in  $((1, 0^{n-1})^P, \mathfrak{X})$  can be added to  $\tilde{H}(n, m+1)$  by Lemma 4.1 and (4.2). Thus we may suppose  $k_0^{(j)} < n$  for each  $j \in \{1, \dots, m\}$ . Moreover the following holds.

**Proposition 4.7.** *Let  $i$  be a positive integer. Suppose  $i$  is even if  $n$  is even, and  $i$  is arbitrary if  $n$  is odd. If each vector in  $\mathfrak{X} \subset \mathbb{R}^{nm}$  can be added to  $\tilde{H}(n, m)$  while maintaining  $m$ -distance, then each vector in  $(\mathfrak{X}, 1^n, \dots, 1^n) \subset \mathbb{R}^{n(m+in)}$  can be added to  $\tilde{H}(n, m+in)$  while maintaining  $m$ -distance.*

*Proof.* Let  $k_i^{(j)}$  be the parameters of  $\mathfrak{X}$ . Since each vector in  $\mathfrak{X}$  can be added to  $\tilde{H}(n, m)$ , the value

$$M_{\mathfrak{X}} = \sum_{j=1}^m \left( 1 - n - 2k_0^{(j)} - \frac{k_0^{(j)2}}{n} + \sum_{i=1}^{t_j} i^2 k_i^{(j)} + 2t_j \right)$$

is an even integer at most  $2m$ . The set  $\mathfrak{Y} = (\mathfrak{X}, 1^n, \dots, 1^n) \subset \mathbb{R}^{n(m+in)}$  satisfies that the value

$$M_{\mathfrak{Y}} = M_{\mathfrak{X}} + i(n-1)$$

is an even integer at most  $2(m+in)$ . This implies the proposition.  $\square$

The following theorem gives the minimum value of  $m$  such that  $\tilde{H}(n, m)$  is not maximal for fixed  $n, l, k_0^{(1)}, \dots, k_0^{(l)}$  in Lemma 4.4.

**Theorem 4.8.** *Let integers  $n, l, k_0^{(1)}, \dots, k_0^{(l)}$  be fixed. Suppose  $k_0^{(j)} < n$  for each  $j \in \{1, \dots, l\}$ . Let*

$$i = \begin{cases} \mathfrak{i} & \text{if } n \text{ is odd,} \\ \mathfrak{i} & \text{if } n \text{ is even and } \mathfrak{i} \text{ is even,} \\ \mathfrak{i} + 1 & \text{if } n \text{ is even and } \mathfrak{i} \text{ is odd,} \end{cases}$$

where

$$\mathfrak{i} = \left\lceil \frac{\sum_{j=1}^l k_0^{(j)}(1 + k_0^{(j)})}{n+1} \right\rceil.$$

If integers  $n, l, k_0^{(1)}, \dots, k_0^{(m)}$  satisfy Lemma 4.4 (2), then the minimum value of  $m$  is

$$l - \sum_{j=1}^l (k_0^{(j)})^2 + in.$$

*Proof.* Since the value

$$M = \sum_{j=1}^l k_0^{(j)} + m + l - \frac{\sum_{j=1}^l (k_0^{(j)})^2 + m - l}{n}$$

is an integer,  $\sum_{j=1}^l (k_0^{(j)})^2 + m - l$  is a multiple of  $n$ . We may express  $m = l - \sum_{j=1}^l (k_0^{(j)})^2 + in$  for some integer  $i$ . Now we have

$$M = \sum_{j=1}^l k_0^{(j)}(1 - k_0^{(j)}) + (n-1)i + 2l.$$

Since  $0 < M \leq 2m = 2(l - \sum_{j=1}^l (k_0^{(j)})^2 + in)$ , it follows that

$$i \geq \left\lceil \frac{\sum_{j=1}^l k_0^{(j)}(1 + k_0^{(j)})}{n+1} \right\rceil.$$

When  $n$  is odd,  $M$  is even. When  $n$  is even,  $M$  is even if and only if  $i$  is even. Therefore the theorem follows from Lemma 4.1.  $\square$



Table 6: Each vector in  $\mathfrak{X}_0$  can be added to  $\tilde{H}(n, m)$ 

$m$	$n$	$n\mathfrak{X}_0$
3	3	$(1^3, 1^3, 1^3), ((2^2, -1)^P, 1^3, 1^3), ((2^2, -1)^P, (2^2, -1)^P, 1^3)$
	5	$((5, 0^4)^P, (2^4, -3^1)^P, 1^5), ((5, 0^4)^P, (3^3, -2^2)^P, 1^5)$
	9	$((4^6, -5^3)^P, 1^9, 1^9), ((5^5, -4^4)^P, 1^9, 1^9)$
	11	$((3^9, -8^2)^P, 1^{11}, 1^{11}), ((8^4, -3^7)^P, 1^{11}, 1^{11})$
4	2	$(1^2, 1^2, 1^2, 1^2)$
	3	$((3, 0^2)^P, 1^3, 1^3, 1^3), ((3, 0^2)^P, (2^2, -1)^P, 1^3, 1^3), ((3, 0^2)^P, (2^2, -1)^P, (2^2, -1)^P, 1^3)$
	5	$((2^4, -3^1)^P, (2^4, -3^1)^P, 1^5, 1^5), ((3^3, -2^2)^P, (2^4, -3^1)^P, 1^5, 1^5),$ $((3^3, -2^2)^P, (3^3, -2^2)^P, 1^5, 1^5), ((5, 0^4)^P, (5, 0^4)^P, (2^4, -3^1)^P, 1^5),$ $((5, 0^4)^P, (5, 0^4)^P, (3^3, -2^2)^P, 1^5)$
	6	$((3^4, -3^2)^P, 1^6, 1^6, 1^6), ((5^2, -1^4)^P, (3^4, -3^2)^P, 1^6, 1^6)$
	7	$((2^6, -5^1)^P, 1^7, 1^7, 1^7), ((5^3, -2^4)^P, 1^7, 1^7, 1^7), ((6^2, -1^5)^P, (2^6, -5^1)^P, 1^7, 1^7),$ $((6^2, -1^5)^P, (5^3, -2^4)^P, 1^7, 1^7)$
	9	$((9, 0^8)^P, (4^6, -5^3)^P, 1^9, 1^9), ((9, 0^8)^P, (5^5, -4^4)^P, 1^9, 1^9)$
	11	$((11, 0^{10})^P, (3^9, -8^2)^P, 1^{11}, 1^{11}), ((11, 0^{10})^P, (8^4, -3^7)^P, 1^{11}, 1^{11})$
	13	$((6^8, -7^5)^P, 1^{13}, 1^{13}, 1^{13}), ((7^7, -6^6)^P, 1^{13}, 1^{13}, 1^{13})$
	14	$((5^{10}, -9^4)^P, 1^{14}, 1^{14}, 1^{14}), ((9^6, -5^8)^P, 1^{14}, 1^{14}, 1^{14})$
	19	$((4^{16}, -15^3)^P, 1^{19}, 1^{19}, 1^{19}), ((15^5, -4^{14})^P, 1^{19}, 1^{19}, 1^{19})$

Table 7:  $\mathfrak{X}_0$  satisfying  $M_{\mathfrak{X}_0} < 2m$ 

$m$	$n$	$n\mathfrak{X}_0$
3	3	$(1^3, 1^3, 1^3), ((2^2, -1)^P, 1^3, 1^3)$
4	2	$(1^2, 1^2, 1^2, 1^2)$
	3	$((3, 0^2)^P, 1^3, 1^3, 1^3), ((3, 0^2)^P, (2^2, -1)^P, 1^3, 1^3)$
	6	$((3^4, -3^2)^P, 1^6, 1^6, 1^6)$
	7	$((2^6, -5^1)^P, 1^7, 1^7, 1^7), ((5^3, -2^4)^P, 1^7, 1^7, 1^7)$

## 5 Largest $m$ -distance sets that contain $\tilde{H}(n, m)$

In this section, we classify the largest  $m$ -distance sets that contain  $\tilde{H}(n, m)$ . For fixed  $m$  and each  $n$  such that  $n \leq m^2 + m - 1$ , we obtain all possible parameters  $(k_0^{(1)}, \dots, k_0^{(m)})$  satisfying Lemma 4.4 (2) by an exhaustive computer search. For  $m = 3, 4$ , Table 6 shows the all sets  $\mathfrak{X}_0$  obtained from the possible parameters  $(k_0^{(1)}, \dots, k_0^{(m)})$ , up to block permutations.

By repeating the inverse modification of  $\mathfrak{X}'$  in Remark 4.3,  $\mathfrak{X}_0$  in Table 6 can be modified to  $\mathfrak{X}$  whose element can be added to  $\tilde{H}(n, m)$ . When  $M_{\mathfrak{X}} < 2m$ , we apply the inverse modification of  $\mathfrak{X}'$  to  $\mathfrak{X}$ . Note that the inverse modification of  $\mathfrak{X}'$  sometimes has several possibilities. The sets  $\mathfrak{X}_0$  with  $M_{\mathfrak{X}_0} < 2m$  are in Table 7 for  $m = 3, 4$ . Table 8 is the list of  $\mathfrak{X}$  obtained from  $\mathfrak{X}_0$ , up to block permutations.

We would like to find the largest sets, which can be added to  $\tilde{H}(n, m)$  while maintaining  $m$ -distance and are in a union of the sets  $\mathfrak{X}$  in Tables 6 and 8. First we classify the largest subsets of  $\mathfrak{X}$  whose maximum distance is at most  $\sqrt{2m}$ . Such subsets can be added to  $\tilde{H}(n, m)$ . The maximum distance of each  $\mathfrak{X}$  in Table 6 is at most  $\sqrt{2m}$  except for the sets in Table 9. The largest subsets of  $\mathfrak{X}$  in Table 9 is at most  $\sqrt{2m}$  can be determined by some results related to

Table 8:  $\mathfrak{X}$  obtained from  $\mathfrak{X}_0$  in Table 7

$m$	$n$	$n\mathfrak{X}$
3	3	$((4, 1, -2)^P, 1^3, 1^3), ((5, -1^2)^P, 1^3, 1^3)$
4	2	$((3, -1)^P, 1^2, 1^2, 1^2)$
	3	$((3, 0^2)^P, (4, 1, -2)^P, 1^3, 1^3), ((3^2, -3)^P, 1^3, 1^3, 1^3),$ $((3, 0^2)^P, (5, -1^2)^P, 1^3, 1^3)$
	6	$((9, 3^2, -3^3)^P, 1^6, 1^6, 1^6)$
	7	$((9, 2^4, -5^2)^P, 1^7, 1^7, 1^7), ((12, 5, -2^5)^P, 1^7, 1^7, 1^7)$

 Table 9:  $\mathfrak{X}(\{k_i^{(j)}\})$  which needs Erdős–Ko–Rado type results

$m$	$n$	$n\mathfrak{X}$	Maximum set $X$	$ X $	reason
3	9	$((5^5, -4^4)^P, 1^9, 1^9)$	$(\mathcal{F}_3(5, 2, 5), 1^9, 1^9)$	56	Theorem 5.1 (2)
	11	$((8^4, -3^7)^P, 1^{11}, 1^{11})$	$(\mathcal{F}_0(4, 1, 8), 1^{11}, 1^{11})$	120	Theorem 5.1 (1)
4	7	$((6^2, -1^5)^P, (5^3, -2^4)^P, 1^7, 1^7)$	$((6^2, -1^5)^P, \mathcal{F}_0(3, 1, 5), 1^7, 1^7)$	315	Lemma 5.4
	9	$((9, 0^8)^P, (5^5, -4^4)^P, 1^9, 1^9)$	$((9, 0^8)^P, \mathcal{F}_3(5, 2, 5), 1^9, 1^9)$	504	Lemma 5.4
	11	$((11, 0^{10})^P, (8^4, -3^7)^P, 1^{11}, 1^{11})$	$((11, 0^{10})^P, \mathcal{F}_0(4, 1, 8), 1^{11}, 1^{11})$	1320	Lemma 5.4
	13	$((6^8, -7^5)^P, 1^{13}, 1^{13}, 1^{13})$	$(\mathcal{F}_4(8, 4, 6), 1^{13}, 1^{13}, 1^{13})$	495	Theorem 5.1 (3)
			$(\mathcal{F}_5(8, 4, 6), 1^{13}, 1^{13}, 1^{13})$	495	Theorem 5.1 (3)
		$((7^7, -6^6)^P, 1^{13}, 1^{13}, 1^{13})$	$(\mathcal{F}_3(7, 3, 7), 1^{13}, 1^{13}, 1^{13})$	372	Theorem 5.1 (2)
	14	$((9^6, -5^8)^P, 1^{14}, 1^{14}, 1^{14})$	$(\mathcal{F}_1(6, 2, 9), 1^{14}, 1^{14}, 1^{14})$	525	Theorem 5.1 (2)
	19	$((15^5, -4^{14})^P, 1^{19}, 1^{19}, 1^{19})$	$(\mathcal{F}_0(5, 1, 15), 1^{19}, 1^{19}, 1^{19})$	3060	Theorem 5.1 (1)

the Erdős–Ko–Rado theorem. Note that  $(\alpha^{k_1}, (\alpha-1)^{k_2})^P$  (*resp.*  $(\alpha^{k_1}, (\alpha-1)^{k_2}, (\alpha-2)^{k_3})^P$ ) is isometric to  $(1^{k_1}, 0^{k_2})^P$  (*resp.*  $(1^{k_1}, 0^{k_2}, -1^{k_3})^P$ ). A subset  $X$  of  $(1^k, 0^{n-k})$  is  $(k-m)$ -*intersecting* if  $d(\mathbf{x}, \mathbf{y})^2 \leq 2m$  for any  $\mathbf{x}, \mathbf{y} \in X$ . A family of non-empty subsets  $X_1, \dots, X_t$  of  $(1^k, 0^{n-k})$  is *cross-intersecting* if  $d(\mathbf{x}, \mathbf{y})^2 < 2k$  for any  $\mathbf{x} \in X_i, \mathbf{y} \in X_j$  and any  $i, j$  with  $i \neq j$ . Let  $\mathcal{F}_r$  denote

$$\mathcal{F}_r = \mathcal{F}_r(k, t) = \{(x_1, \dots, x_n) \in (1^k, 0^{n-k})^P : |\{i \in \{1, \dots, t+2r\} : x_i = 1\}| \geq t+r\}$$

for  $n \geq k \geq t+r$  and  $n \geq t+2r$ . Note that  $\mathcal{F}_0 = (1^t, (1^{k-t}, 0^{n-k})^P)$ . We collect Erdős–Ko–Rado type results that are needed later.

**Theorem 5.1** ([7, 16, 2]). *If  $X \subset (1^k, 0^{n-k})$  is  $t$ -intersecting, then the following hold.*

- (1) *If  $n > (k-t+1)(t+1)$ , then  $|X| \leq \binom{n-t}{k-t}$ , and the largest set is  $\mathcal{F}_0$ , up to permutations of coordinates.*
- (2) *If  $(k-t+1)(2+(t-1)/(r+1)) < n < (k-t+1)(2+(t-1)/r)$  for some  $r \in \mathbb{N}$ , then  $|X| \leq |\mathcal{F}_r|$ , and the largest set is  $\mathcal{F}_r$ , up to permutations of coordinates.*
- (3) *If  $(k-t+1)(2+(t-1)/(r+1)) = n$  for some  $r \in \mathbb{N}$ , then  $|X| \leq |\mathcal{F}_r| = |\mathcal{F}_{r+1}|$ , and the largest set is  $\mathcal{F}_r$  or  $\mathcal{F}_{r+1}$ , up to permutations of coordinates.*

**Theorem 5.2** ([10]). *Suppose  $n \geq 2k$ . If a pair of subsets  $X, Y$  of  $(1^k, 0^{n-k})$  is cross-intersecting, then*

$$|X| + |Y| \leq \binom{n}{k} - \binom{n-k}{k} + 1.$$

**Theorem 5.3** ([9, 5]). *Suppose  $n > 2k$  and  $s > n/k$ . If a family of subsets  $X_1, \dots, X_s$  of  $(1^k, 0^{n-k})$  is cross-intersecting, then*

$$\sum_{i=1}^s |X_i| \leq s \binom{n-1}{k-1}.$$

*If equality holds, then  $X_i = \mathcal{F}_0$  for each  $i \in \{1, \dots, s\}$ , up to permutations of coordinates.*

We use the notation

$$\mathcal{F}_r(k, t, k_0) = \{\mathbf{x} \in (k_0^k, (k_0-n)^{n-k})^P : 1/n(\mathbf{x} - (k_0-n)\mathbf{1}) \in \mathcal{F}_r(k, t)\},$$

where  $\mathbf{1} = (1^n)$ . By Theorem 5.1, we can determine the largest subsets as Table 9 except for  $(m, n) = (4, 7), (4, 9), (4, 11)$ .

**Lemma 5.4.** *If  $X$  is the largest subset of  $\mathfrak{X}_0$  whose distances are in  $\{\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}\}$ , then the following hold.*

- (1) *For  $\mathfrak{X}_0 = (1/7)((6^2, -1^5)^P, (5^3, -2^4)^P, 1^7, 1^7)$ , we have*

$$X = \left( \left( \frac{6^2}{7}, -\frac{1^5}{7} \right)^P, \frac{1}{7}\mathcal{F}_0(3, 1, 5), \frac{1^7}{7}, \frac{1^7}{7} \right),$$

*up to permutations on coordinates in the second block.*

(2) For  $\mathfrak{X}_0 = (1/9)((9, 0^8)^P, (5^5, -4^4)^P, 1^9, 1^9)$ , we have

$$X = \left( (1^1, 0^8)^P, \frac{1}{9} \mathcal{F}_3(5, 2, 5), \frac{1^9}{9}, \frac{1^9}{9} \right),$$

up to permutations on coordinates in the second block.

(3) For  $\mathfrak{X}_0 = (1/11)((11, 0^{10})^P, (8^4, -3^7)^P, 1^{11}, 1^{11})$ , we have

$$X = \left( (1^1, 0^{10})^P, \frac{1}{11} \mathcal{F}_0(4, 1, 8), \frac{1^{11}}{11}, \frac{1^{11}}{11} \right),$$

up to permutations on coordinates in the second block.

*Proof.* We use the notation  $S_{\mathbf{a}} = \{\mathbf{x}_2 \mid \mathbf{x}_1 = \mathbf{a}, (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \in X\}$ , where  $\mathbf{x}_i$  is a vector in  $\mathbb{R}^n$ .

(1): If  $d(\mathbf{x}_1, \mathbf{y}_1)^2 = 4$ , then  $d(\mathbf{x}_2, \mathbf{y}_2)^2 \leq 4$  for  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4), (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) \in X$ . Thus we have  $d(\mathbf{a}, \mathbf{b})^2 \leq 4$  for any  $\mathbf{a} \in S_{\mathbf{x}_1}$ ,  $\mathbf{b} \in S_{\mathbf{y}_1}$  such that  $d(\mathbf{x}_1, \mathbf{y}_1) = 4$ , and hence a pair  $\{S_{\mathbf{x}_1}, S_{\mathbf{y}_1}\}$  is cross-intersecting. The set  $((6/7)^2, (-1/7)^5)^P$  is isometric to  $(1^2, 0^5)^P$ . The set  $(1^2, 0^5)^P$  has a triangle decomposition  $\{T_i\}_{0 \leq i \leq 6}$ , for example

$$T_i = \{(0, 0, 1, 0, 1, 0, 0)^{\sigma(i)}, (0, 1, 0, 0, 0, 1, 0)^{\sigma(i)}, (1, 0, 0, 0, 0, 0, 1)^{\sigma(i)}\},$$

where  $(x_1, \dots, x_7)^{\sigma(i)} = (x_{1+i}, \dots, x_{7+i})$  such that the indices are in  $\mathbb{Z}/7\mathbb{Z}$ . By Theorems 5.2 and 5.3, we have

$$\sum_{\mathbf{a} \in T_i} |S_{\mathbf{a}}| \leq \begin{cases} \binom{7}{3} = 35, & \text{if two sets are empty,} \\ \binom{7}{3} - \binom{4}{3} + 1 = 32, & \text{if only one set is empty,} \\ 3\binom{6}{2} = 45, & \text{if no set is empty.} \end{cases}$$

It therefore follows that

$$|X| = \sum_{i=0}^6 \sum_{\mathbf{a} \in T_i} |S_{\mathbf{a}}| \leq 7 \cdot 45 = 315.$$

If equality holds, then  $X$  is the set defined in (1).

(2): The second block can be identified with  $(1^4, 0^5)^P$ , which is isometric to  $(1^5, 0^4)^P$ . For distinct vectors  $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4), (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) \in X$ , we have  $d(\mathbf{x}_1, \mathbf{y}_1)^2 = 2$  and  $d(\mathbf{x}_2, \mathbf{y}_2)^2 \leq 6$ . Thus a family  $\{S_{\mathbf{x}_1}\}_{\mathbf{x}_1 \in X}$  can be interpreted as a cross-intersecting family. The statement (2) therefore follows from Theorem 5.3.

(3): This case is proved by a similar manner to (2). □

For  $\mathfrak{X}$  in Table 8, the largest subsets of  $\mathfrak{X}$  whose maximum distance is at most  $\sqrt{2m}$  are in Table 10. By computer search using Magma, we classify maximal cliques in the graph  $(\mathfrak{X}, E)$ , where  $E = \{(\mathbf{x}, \mathbf{y}) \mid d(\mathbf{x}, \mathbf{y})^2 \leq 2m\}$  except for  $n = 7$ . We use the following results for  $n = 7$ .

**Proposition 5.5** ([1]). *Assume  $m + k < l$ . If  $X \subset (1^m, 0^k, -1^l)^P$  has maximum distance smaller than that of  $(1^m, 0^k, -1^l)^P$ , then*

$$|X| \leq \binom{n-1}{m+k-1} \binom{m+k}{m}.$$

*The largest set is  $(1, (1^{m-1}, 0^k, -1^l)^P) \cup (0, (1^m, 0^{k-1}, -1^l)^P)$ , up to permutations of coordinates.*

Table 10: Largest subset whose distance at most  $\sqrt{2m}$ 

$m$	$n$	$n\mathfrak{X}$	largest subset $X$	$ X $
3	3	$((4, 1, -2)^P, 1^3, 1^3)$	$\{((u_1, 1, v_1), 1^3, 1^3), ((u_2, v_2, 1), 1^3, 1^3), ((1, u_3, v_3), 1^3, 1^3)\} ((u_i, v_i) = (4, -2) \text{ or } (-2, 4))$	3
		$((5, -1^2)^P, 1^3, 1^3)$	$\{((5, -1^2), 1^3, 1^3)\}$	1
4	2	$((3, -1)^P, 1^2, 1^2, 1^2)$	$((3, -1)^P, 1^2, 1^2, 1^2)$	2
	3	$((3, 0^2)^P, (4, 1, -2)^P, 1^3, 1^3)$	$\{((3, 0^2)^P, (u_1, 1, v_1), 1^3, 1^3), ((3, 0^2)^P, (u_2, v_2, 1), 1^3, 1^3), ((3, 0^2)^P, (1, u_3, v_3), 1^3, 1^3)\} ((u_i, v_i) = (4, -2) \text{ or } (-2, 4))$	9
		$((3^2, -3)^P, 1^3, 1^3, 1^3)$	$((3^2, -3)^P, 1^3, 1^3, 1^3)$	3
		$((3, 0^2)^P, (5, -1^2)^P, 1^3, 1^3)$	$((3, 0^2), (5, -1^2)^P, 1^3, 1^3), ((3, 0^2)^P, (5, -1^2), 1^3, 1^3)$	3
	6	$((9, 3^2, -3^3)^P, 1^6, 1^6, 1^6)$	$(9, (3^2, -3^3)^P) \cup (3^2, (9, -3^3)^P) \cup (3, 9, (3, -3^3)^P)$	18
	7	$((9, 2^4, -5^2)^P, 1^7, 1^7, 1^7)$	$X_7(9), Y_7(9), Z_7(9)$	37
		$((12, 5, -2^5)^P, 1^7, 1^7, 1^7)$	$(12, (5, -2^5)^P) \cup (5, (12, -2^5)^P)$	12

Let  $X_1 = (1, (0, -1^2)^P) \cup (0, (1, -1^2)^P)$ ,  $Y_1 = (-1, (1, 0, -1)^P)$ , and  $Z_2 = (-1, (1, 0^2, -1)^P)$ . We inductively define

$$\begin{aligned} X_k &= (0, X_{k-1}) \cup (1, (0^k, -1^2)^P) \quad (k \geq 2), \\ Y_k &= (0, Y_{k-1}) \cup (1, (0^k, -1^2)^P) \quad (k \geq 2), \\ Z_k &= (0, Z_{k-1}) \cup (1, (0^k, -1^2)^P) \quad (k \geq 3). \end{aligned}$$

**Theorem 5.6** ([1]). *If  $X \subset (1, 0^k, -1^2)^P$  has maximum distance smaller than that of  $(1, 0^k, -1^2)^P$ , then*

$$|X| \leq \binom{k+3}{3} + 2.$$

*The largest sets are  $X_k$  ( $k \geq 1$ ),  $Y_k$  ( $k \geq 1$ ), and  $Z_k$  ( $k \geq 2$ ), up to permutations of coordinates.*

Let  $X_k(k_0)$  be the subset of  $(k_0, (k_0 - n)^k, (k_0 - 2n)^2)$  obtained from  $X_k$  by replacing the entries 1, 0,  $-1$  to  $k_0$ ,  $k_0 - n$ ,  $k_0 - 2n$ , respectively. The sets  $Y_k(k_0)$ ,  $Z_k(k_0)$  are similarly defined.

Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be distinct sets in Tables 6 and 8. We consider when  $\mathbf{x} \in \mathfrak{X}$ ,  $\mathbf{y} \in \mathfrak{Y}$  can be simultaneously added to  $\tilde{H}(n, m)$ . An element  $(x_{11}, \dots, x_{1n}, \dots, x_{m1}, \dots, x_{mn})$  of  $\mathfrak{X}$  is *canonical* if  $x_{i,l} \geq x_{i,l+1}$  for any  $i \in \{1, \dots, m\}$ ,  $l \in \{1, \dots, n-1\}$ .

**Lemma 5.7.** *If  $\mathfrak{X}$  and  $\mathfrak{Y}$  are distinct sets that are expressed as (4.1), then the following are equivalent.*

- (1) *There exist  $\mathbf{x} \in \mathfrak{X}$ ,  $\mathbf{y} \in \mathfrak{Y}$  such that  $d(\mathbf{x}, \mathbf{y})^2 \in \{2, 4, \dots, 2m\}$ .*
- (2) *For canonical elements  $\mathbf{x}' \in \mathfrak{X}$ ,  $\mathbf{y}' \in \mathfrak{Y}$ , we have  $d(\mathbf{x}', \mathbf{y}')^2 \in \{2, 4, \dots, 2m\}$ .*

Moreover if

$$\max_{\mathbf{x} \in \mathfrak{X}, \mathbf{y} \in \mathfrak{Y}} d(\mathbf{x}, \mathbf{y})^2 \in \{2, 4, \dots, 2m\},$$

then any  $\mathbf{x} \in \mathfrak{X}$ ,  $\mathbf{y} \in \mathfrak{Y}$  satisfy  $d(\mathbf{x}, \mathbf{y})^2 \in \{2, 4, \dots, 2m\}$ .

*Proof.* (2)  $\Rightarrow$  (1) is clear. We prove (1)  $\Rightarrow$  (2). Suppose  $d(\mathbf{x}, \mathbf{y})^2 \in \{2, 4, \dots, 2m\}$  for some  $\mathbf{x} \in \mathfrak{X}$ ,  $\mathbf{y} \in \mathfrak{Y}$ . By permutation of coordinates, we may suppose  $\mathbf{x}$  is canonical. When there exist  $i, l$  such that  $y_{i,l} < y_{i,l+1}$ , we switch the positions of  $y_{i,l}$ ,  $y_{i,l+1}$  to reduce the distance between  $\mathbf{x}$  and  $\mathbf{y}$ . Indeed, letting  $x_{i,l} = \alpha$ ,  $x_{i,l+1} = \alpha - s$ ,  $y_{i,l} = \beta$ ,  $y_{i,l+1} = \beta + t$ , where  $s, t$  are non-negative integers, the difference of the squared distances is

$$((x_{i,l} - y_{i,l})^2 + (x_{i,l+1} - y_{i,l+1})^2) - ((x_{i,l} - y_{i,l+1})^2 + (x_{i,l+1} - y_{i,l})^2) = 2st \geq 0. \quad (5.1)$$

By repeating this modification, we can obtain canonical element  $\mathbf{y}$  that satisfies  $d(\mathbf{x}, \mathbf{y})^2 \in \{2, 4, \dots, 2m\}$ .

The second assertion is clear by (5.1).  $\square$

Let  $V(n, m)$  be the family consisting of all sets  $\mathfrak{X}$  whose element can be added to  $\tilde{H}(n, m)$ . Let  $E(n, m) = \{(\mathfrak{X}, \mathfrak{Y}) : d(x', y') \in \{2, 4, \dots, 2m\}\} \subset V(n, m) \times V(n, m)$ , where  $x', y'$  are the canonical elements of  $\mathfrak{X}, \mathfrak{Y}$ , respectively. Let  $\mathcal{G}(n, m)$  be the graph  $(V(n, m), E(n, m))$ . By computer search using Magma, we can classify maximal cliques in  $\mathcal{G}(n, m)$ , up to block permutations. Table 11 shows the all maximal cliques in  $\mathcal{G}(n, m)$ . By Lemma 5.7, for any distinct sets  $\mathfrak{X}, \mathfrak{Y}$  in a clique in Table 11, any  $\mathbf{x} \in \mathfrak{X}$ ,  $\mathbf{y} \in \mathfrak{Y}$  can be simultaneously added to  $\tilde{H}(n, m)$ , except for the combination of  $((3, 0^2)^P, 1^3, 1^3, (4, 1, -2)^P)$  and  $(1^3, (3, 0^2)^P, 1^3, (5, -1^2)^P)$ . We replace  $\mathfrak{X}$  in a clique in Table 11 with one of the largest subsets of  $\mathfrak{X}$ , which can be added to  $\tilde{H}(n, m)$ . Finally we obtain Table 12, which includes the all largest subsets that can be added to  $\tilde{H}(n, m)$ .

## 6 Maximal 2-distance sets that are in $\mathbb{R}^{2n-1}$ and contain $\tilde{H}(n, 2)$

The Hamming graph  $H(n, 2)$  is embedded into  $\mathbb{R}^{2(n-1)}$  as a 2-distance set. Indeed the representation is

$$\tilde{H}(n, 2) = ((1^1, 0^{n-1})^P, (1^1, 0^{n-1})^P) \subset \{\mathbf{x} \in \mathbb{R}^{2n} \mid \sum_{i=1}^n x_i = 1, \sum_{i=n+1}^{2n} x_i = 1\} \cong \mathbb{R}^{2(n-1)},$$

where  $\cong$  means isometry. In this section, we identify  $\tilde{H}(n, 2)$  with a 2-distance set in  $\mathbb{R}^{2n-1}$  as follows

$$\begin{aligned} \tilde{H}(n, 2) &\cong \hat{H}(n, 2) = ((1^1, 0^{n-1})^P, (1^1, 0^{n-1})^P, 0) \\ &\subset \{\mathbf{x} \in \mathbb{R}^{2n+1} \mid \sum_{i=1}^n x_i = 1, \sum_{i=n+1}^{2n} x_i = 1\} \cong \mathbb{R}^{2n-1}. \end{aligned}$$

We consider maximal 2-distance sets that are in  $\mathbb{R}^{2n-1}$  and contain  $\hat{H}(n, 2)$ . Bannai, Sato, and Shigezumi [4] considered a similar problem for the Johnson graph  $J(n, 2)$ , and classified maximal 2-distance sets.

Suppose  $\mathbf{x} \in \mathbb{R}^{2n-1}$  can be added to  $\hat{H}(n, 2)$  while maintaining 2-distance. By a similar argument to that in Section 2,  $\mathbf{x}$  forms

$$\mathbf{x} \in \left( \left( \left( \frac{k}{n} \right)^{n-k+1}, \left( \frac{k}{n} - 1 \right)^{k-1} \right)^P, \frac{1}{n} \right)^P, \beta$$

Table 11: Maximal cliques in  $\mathcal{G}(n, m)$ 

$m$	$n$	maximal clique
3	3	$((5, -1^2)^P, 1^3, 1^3), (1^3, (2^2, -1)^P, (2^2, -1)^P), ((2^2, -1)^P, 1^3, 1^3)$ $(1^3, 1^3, 1^3), ((4, 1, -2)^P, 1^3, 1^3), (1^3, (4, 1, -2)^P, 1^3), (1^3, 1^3, (4, 1, -2)^P)$
		$((3^3, -2^2)^P, 1^5, (5, 0^4)^P), (1^5, (2^4, -3)^P, (5, 0^4)^P)$
	5	$((5^5, -4^4)^P, 1^9, 1^9)$ $((4^6, -5^3)^P, 1^9, 1^9), (1^9, (4^6, -5^3)^P, 1^9), (1^9, (4^6, -5^3)^P, 1^9)$
		$((3^9, -8^2)^P, 1^{11}, 1^{11})$ $((8^4, -3^7)^P, 1^{11}, 1^{11})$
	11	
4	2	$(1^2, 1^2, 1^2, 1^2), ((3, -1)^P, 1^2, 1^2, 1^2), (1^2, (3, -1)^P, 1^2, 1^2),$ $(1^2, 1^2, (3, -1)^P, 1^2), (1^2, 1^2, 1^2, (3, -1)^P)$
		$((3, 0^2)^P, 1^3, 1^3, 1^3), (1^3, (3, 0^2)^P, 1^3, (2^2, -1)^P),$ $(1^3, (2^2, -1)^P, (3, 0^2)^P, 1^3), (1^3, 1^3, (2^2, -1)^P, (3, 0^2)^P),$ $((2^2, -1)^P, (3, 0^2)^P, (2^2, -1)^P, 1^3), ((2^2, -1)^P, 1^3, (3, 0^2)^P, (2^2, -1)^P),$ $((2^2, -1)^P, (2^2, -1)^P, 1^3, (3, 0^2)^P), ((3^2, -3)^P, 1^3, 1^3, 1^3),$ $((3, 0^2)^P, (4, 1, -2)^P, 1^3, 1^3), (1^3, (5, -1^2)^P, (3, 0^2)^P, 1^3),$ $((3, 0^2)^P, 1^3, (4, 1, -2)^P, 1^3), (1^3, 1^3, (5, -1^2)^P, (3, 0^2)^P),$ $((3, 0^2)^P, 1^3, 1^3, (4, 1, -2)^P), (1^3, (3, 0^2)^P, 1^3, (5, -1^2)^P)$
	5	$(1^5, 1^5, (2^4, -3)^P, (2^4, -3)^P), ((3^3, -2^2)^P, 1^5, 1^5, (2^4, -3)^P),$ $(1^5, (3^3, -2^2)^P, (2^4, -3)^P, 1^5), ((3^3, -2^2)^P, (3^3, -2^2)^P, 1^5, 1^5),$ $((5, 0^4)^P, 1^5, (5, 0^4)^P, (2^4, -3)^P), (1^5, (5, 0^4)^P, (2^4, -3)^P, (5, 0^4)^P),$ $((5, 0^4)^P, (3^3, -2^2)^P, (5, 0^4)^P, 1^5), ((3^3, -2^2)^P, (5, 0^4)^P, 1^5, (5, 0^4)^P)$
		$((3^4, -3^2)^P, 1^6, 1^6, 1^6), (1^6, (5^2, -1^4)^P, 1^6, (3^4, -3^2)^P),$ $(1^6, (3^4, -3^2)^P, (5^2, -1^4)^P, 1^6), (1^6, 1^6, (3^4, -3^2)^P, (5^2, -1^4)^P),$ $((9, 3^2, -3^3)^P, 1^6, 1^6, 1^6)$
	7	$((2^6, -5)^P, 1^7, 1^7, 1^7), (1^7, (6^2, -1^5)^P, (5^3, -2^4)^P, 1^7),$ $(1^7, 1^7, (6^2, -1^5)^P, (5^3, -2^4)^P), (1^7, (5^3, -2^4)^P, 1^7, (6^2, -1^5)^P),$ $((9, 2^4, -5^2)^P, 1^7, 1^7, 1^7),$ $((5^3, -2^4)^P, 1^7, 1^7, 1^7), (1^7, (6^2, -1^5)^P, (2^6, -5)^P, 1^7),$ $(1^7, 1^7, (6^2, -1^5)^P, (2^6, -5)^P), (1^7, (2^6, -5)^P, 1^7, (6^2, -1^5)^P),$ $((12, 5, -2^5)^P, 1^7, 1^7, 1^7)$
		$((9, 0^8)^P, (5^5, -4^4)^P, 1^9, 1^9), (1^9, 1^9, (9, 0^8)^P, (5^5, -4^4)^P)$ $((9, 0^8)^P, (4^6, -5^3)^P, 1^9, 1^9), ((9, 0^8)^P, 1^9, (4^6, -5^3)^P, 1^9),$ $((9, 0^8)^P, 1^9, 1^9, (4^6, -5^3)^P)$
	9	
	11	$((11, 0^{10})^P, (8^4, -3^7)^P, 1^{11}, 1^{11}), (1^{11}, 1^{11}, (11, 0^{10})^P, (3^9, -8^2)^P)$
	13	$((6^8, -7^5)^P, 1^{13}, 1^{13}, 1^{13})$ $((7^7, -6^6)^P, 1^{13}, 1^{13}, 1^{13})$
		$((5^{10}, -9^4)^P, 1^{14}, 1^{14}, 1^{14})$ $((9^6, -5^8)^P, 1^{14}, 1^{14}, 1^{14})$
	14	
	19	$((4^{16}, -15^3)^P, 1^{19}, 1^{19}, 1^{19})$ $((15^5, -4^{19})^P, 1^{19}, 1^{19}, 1^{19})$

Table 12: Sets obtained from a maximal clique in  $\mathcal{G}(n, m)$ 

$m$	$n$	$X$	$ X $	$M$
3	3	$((5, -1^2), 1^3, 1^3) \cup (1^3, (2^2, -1)^P, (2^2, -1)^P) \cup ((2^2, -1)^P, 1^3, 1^3)$	13	40
		$(1^3, 1^3, 1^3) \cup ((4, 1, -2)^C, 1^3, 1^3)^P$	10	37
	5	$((3^3, -2^2)^P, 1^5, (5, 0^4)^P) \cup (1^5, (2^4, -3)^P, (5, 0^4)^P)$	75	200
	9	$(\mathcal{F}_3(5, 2, 5), 1^9, 1^9)$	84	785
		$((4^6, -5^3)^P, 1^9, 1^9)^P$	252	981
	11	$((3^9, -8^2)^P, 1^{11}, 1^{11})$	55	1386
		$(\mathcal{F}_0(4, 1, 8), 1^{11}, 1^{11})$	120	1451
4	2	$(1^2, 1^2, 1^2, 1^2) \cup ((3, -1)^P, 1^2, 1^2, 1^2)^P$	9	25
	3	$((3, 0^2)^P, 1^3, 1^3, 1^3) \cup (1^3, ((3, 0^2)^P, 1^3, (2^2, -1)^P)^C)$ $\cup ((2^2, -1)^P, ((3, 0^2)^P, (2^2, -1)^P, 1^3)^C) \cup (3^2, -3)^P, 1^3, 1^3, 1^3)$ $\cup ((3, 0^2)^P, (\{-2, 4, 1\}, (-2, 1, 4)\}, 1^3, 1^3)^C) \cup (1^3, ((5, -1^2), (3, 0^2)^P, 1^3)^C)$	141	222
		$((3, 0^2)^P, 1^3, 1^3, 1^3) \cup (1^3, ((3, 0^2)^P, 1^3, (2^2, -1)^P)^C)$ $\cup ((2^2, -1)^P, ((3, 0^2)^P, (2^2, -1)^P, 1^3)^C) \cup (3^2, -3)^P, 1^3, 1^3, 1^3)$ $\cup ((3, 0^2)^P, ((4, -2, 1)^C, 1^3, 1^3)^C)$	141	222
	5	$(1^5, 1^5, (2^4, -3)^P, (2^4, -3)^P) \cup ((3^3, -2^2)^P, 1^5, 1^5, (2^4, -3)^P)^{(12)(34)}$ $\cup ((3^3, -2^2)^P, (3^3, -2^2)^P, 1^5, 1^5) \cup ((5, 0^4)^P, 1^5, (5, 0^4)^P, (2^4, -3)^P)^{(12)(34)}$ $\cup ((5, 0^4)^P, (3^3, -2^2)^P, (5, 0^4)^P, 1^5)^{(12)(34)}$	975	1600
	6	$((3^4, -3^2)^P, 1^6, 1^6, 1^6) \cup (1^6, ((5^2, -1^4)^P, 1^6, (3^4, -3^2)^P)^C)$ $\cup ((9, (3^2, -3^3)^P) \cup (3^2, (9, -3^3)^C) \cup (3, 9, (3, -3^3)^C), 1^6, 1^6, 1^6)$	708	2004
	7	$((2^6, -5)^P, 1^7, 1^7, 1^7) \cup (1^7, ((6^2, -1^5)^P, \mathcal{F}_0(3, 1, 5), 1^7)^C)$ $\cup (\{X_7(9), Y_7(9) \text{ or } Z_7(9)\}, 1^7, 1^7, 1^7)$	989	3390
		$((5^3, -2^4)^P, 1^7, 1^7, 1^7) \cup (1^7, ((6^2, -1^5)^P, (2^6, -5)^P, 1^7)^C)$ $\cup (12, (5, -2^5)^P) \cup (5, (12, -2^5)^P)$	488	2889
	9	$((9, 0^8)^P, \mathcal{F}_3(5, 2, 5), 1^9, 1^9), (1^9, 1^9, (9, 0^8)^P, \mathcal{F}_3(5, 2, 5))$	1008	7569
		$((9, 0^8)^P, ((4^6, -5^3)^P, 1^9, 1^9)^C)$	2268	8829
	11	$((11, 0^{10})^P, \mathcal{F}_0(4, 1, 8), 1^{11}, 1^{11}) \cup (1^{11}, 1^{11}, (11, 0^{10})^P, (3^9, -8^2)^P)$	1925	16566
	13	$(\mathcal{F}_4(8, 4, 6), 1^{13}, 1^{13}, 1^{13})$	495	29056
		$(\mathcal{F}_3(7, 3, 7), 1^{13}, 1^{13}, 1^{13})$	372	28933
	14	$((5^{10}, -9^4)^P, 1^{14}, 1^{14}, 1^{14})$	1001	39417
		$(\mathcal{F}_1(6, 2, 9), 1^{14}, 1^{14}, 1^{14})$	525	38941
	19	$((4^{16}, -15^3)^P, 1^{19}, 1^{19}, 1^{19})$	969	131290
		$(\mathcal{F}_0(5, 1, 15), 1^{19}, 1^{19}, 1^{19})$	3060	133381

$$(x_1, \dots, x_n)^C = \{(x_{1+i}, \dots, x_{n+i}) \mid i \in \mathbb{Z}/n\mathbb{Z}\}, \quad (X_1, \dots, X_n)^C = \bigcup_{i \in \mathbb{Z}/n\mathbb{Z}} (X_{1+i}, \dots, X_{n+i}),$$

$$(X_1, X_2, X_3, X_4)^{(12)(34)} = (X_1, X_2, X_3, X_4) \cup (X_2, X_1, X_4, X_3).$$



for some  $\beta \in \mathbb{R}$ , and  $1 \leq k \leq n$ . For  $\mathbf{y} \in \hat{H}(n, 2)$ , we have

$$d(\mathbf{x}, \mathbf{y})^2 = -1 - \frac{1}{n} + k - \frac{k^2}{n} + 2l_1 + 4l_2 + \beta^2 \in \{2, 4\},$$

where  $l_i = |\{q: y_q = 1, x_q = k/n + 1 - i\}| \in \{0, 1\}$ . It therefore holds that

$$\beta = \pm \sqrt{1 + \frac{1}{n} - k + \frac{k^2}{n}}.$$

We use the notation

$$\begin{aligned} X_k^\pm &= \left( \left( \left( \frac{k^{n-k+1}}{n}, \left( \frac{k}{n} - 1 \right)^{k-1} \right)^P, \frac{1}{n} \right)^P, \pm \sqrt{1 + \frac{1}{n} - k + \frac{k^2}{n}} \right), \\ Y_k^\pm &= \left( \left( \frac{k^{n-k+1}}{n}, \left( \frac{k}{n} - 1 \right)^{k-1} \right)^P, \frac{1}{n}, \pm \sqrt{1 + \frac{1}{n} - k + \frac{k^2}{n}} \right), \\ Z_k^\pm &= \left( \frac{1}{n}, \left( \frac{k^{n-k+1}}{n}, \left( \frac{k}{n} - 1 \right)^{k-1} \right)^P, \pm \sqrt{1 + \frac{1}{n} - k + \frac{k^2}{n}} \right) \end{aligned}$$

for  $k \in \{1, \dots, n\}$ . Since the radicand of  $\beta$  is not negative, an element of  $X_k^\pm$  can be added to  $\hat{H}(n, 2)$  only for  $(k, n)$  such that  $k \geq 1$  and  $n \leq 5$ , and for  $(k, n)$  such that  $k \in \{1, n-1, n, n+1\}$  and  $n \geq 6$ . The following is the classification of the maximal sets, which can be added to the corresponding  $\hat{H}(n, 2)$  and has size at least 2.

- $n \geq 3$

- (1)  $X_{n-1}^+ \text{ or } X_{n-1}^-$  [ $n(n-1)$  points]

- $n \geq 2$

- (1)  $Y_n^+, Y_n^-, Z_n^+, \text{ or } Z_n^-$  [ $n$  points]

- $n = 2$

- (1)  $X_1^+ \cup X_1^-$  [2 points]

- $n = 4$

- (1)  $X_1^+ \cup X_1^-$  [2 points]

- (2)  $Y_2^+, Y_2^-, Z_2^+, \text{ or } Z_2^-$  [4 points]

- (3)  $X \subset X_3^+ \cup X_3^-$  [12 points]

$X$  contains only one of  $(\mathbf{x}, \sqrt{1/2})$  or  $(\mathbf{y}, -\sqrt{1/2})$  such that  $x_i = 3/4 \Leftrightarrow y_i = -1/4$ , and  $x_i = -1/4 \Leftrightarrow y_i = 3/4$ .

- $n = 5$

- (1)  $Y_2^+ \cup Z_3^+, \text{ or } Z_2^+ \cup Y_3^+$  [15 points]

- $n = 8$

(1)  $X_1^+ \cup X_9^-$  or  $X_1^- \cup X_9^+$  [2 points]

For  $n \geq 3$  (1),  $X_{n-1}^+ \cup \hat{H}(n, 2)$  is isometric to  $\tilde{J}(2n, 2)$ . For  $n \geq 2$  (1), the graph obtained from  $Y_n^+ \cup \hat{H}(n, 2)$  is bi-regular. For  $n = 5$  (1),  $Y_2^+ \cup Z_3^+ \cup \hat{H}(n, 2)$  is still in  $\mathbb{R}^8$ , and it is also maximal in  $\mathbb{R}^9$ .

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